Plane waves in noncommutative fluids

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Abstract

We study the dynamics of the noncommutative fluid in the Snyder space from the perturbative point of view. To this end, the relevant quantities are treated as series in powers of the noncommutative parameter. At the Planck scale, the relevant terms are of first order and the dynamics is described by a system of coupled linear partial differential equations in which the variables are the fluid density and the fluid potentials. We show that these equations admit a set of solutions that are monocromatic plane waves for the fluid density and two of the potentials and linear for the third potential. The energy-momentum tensor of these solutions is calculated.

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1 Introduction

One of the most fundamental questions that can be posed in the context of the noncommutative space-time paradigm [1, 2] is: how are the noncommutative properties of the underlying space reflected in the phenomenological equations of continuous distributions of matter? Most of the efforts dedicated to the modelling of continuous systems in noncommutative spaces have been devoted to noncommutative field theories that at large scales compared to the relevant noncommutative parameter of the space-time coincide with classical field theories [3, 4]. However, many of the known continuos systems in commutative spaces are described by fluid theories. Therefore, one can rephrase the above question in terms of fluid distributions of matter.

The interest in formulating a fluid theory in noncommutative spaces is note entirely academic. Indeed, recent studies have shown that there are systems that display both fluid and noncommutative properties simultaneously. For example, the comparision of the symmetries of the noncommutative field theories with the transformations of the fluid phase space in commutative space-time shows an analogy between the symplectic preserving diffeomorphism of the noncommutative spaces and the volume preserving diffeomorphism of the commutative phase space. This analogy suggests that there should be a noncommutative analogoue of the Bernoulli equation [5, 6, 7, 8]. By exploiting the similarity between the two spaces, the symplectic structure of the rotational and irrotational fluids has been generalized to noncommutative spaces [9]. Another concrete example of the occurence of both fluidity and noncommutativity in physical systems is given by the quantum Hall liquid in which the granularity property can be described in terms of noncommutative gauge fields [10, 11, 12, 13, 14, 15]. Other systems in which the two properties have shown up together are the charged particles studied in [16] and the linear cosmological perturbations of quantum fluids from [17].

The construction of noncommutative fluid models is not a trivial task. Indeed, the fluid equations of motion in commutative spaces are established from phenomenological considerations based on the behaviour of the long range degrees of freedom of statistical systems. In general, the definition and the interpretation of the fluid quantities in terms of noncommutative spaces is very difficult. A more direct approach would be to derive the fluid properties from their microscopic degrees of freedom. However, the formulation and the understanding of the statistical mechanics in noncommutative spaces is still an open problem (see for tentative approaches [18, 19]). Nevertheless, there is different approach to the commutative perfect relativistic fluids based on field theoretic methods. In this setting, the fluid dynamics is described by the Euler-Lagrange equations obtained from a classical first-order Lagrangian functional defined on the space of fluid degrees of freedom. These are called fluid potentials and they parametrize the fluid velocity field [20]. The choice of the fluid potentials is not unique and, in general, it is made with the following idea in mind. In general, the construction of a Lagrangian and Hamiltonian formalism for rotational relativistic ideal fluid is obstructed by Casimir-like invariants that prevent finding the inverse of the symplectic form defined in the phase space of the degrees of freedom of the fluid. However, by parametrizing the velocity in terms of fluid potentials such that the invariant be given by a surface integral, the obstruction can be removed. The choice of the fluid potentials is not unique. When it is made in terms of real functions $\theta(x)$, $\alpha(x)$ and $\beta(x)$ it is called the Clebsch parametrization [20, 21, 22] while the fluid potentials given in terms of one real $\theta(x)$ and two complex functions $z(x)$ and $\bar{z}(x)$, respectively, define the so called Kähler parametrization [23, 25, 26, 27, 28, 29, 30].

The last approach to the dynamics of the relativistic ideal fluid in terms of the fluid potentials is suited to constructing a fluid model in noncommutative spaces. By following this line

of thought, some of us have obtained for the first time a large class of noncommutative fluids in canonical noncommutative spaces, i. e. spaces with the coordinate algebra characterized by a constant antisymmetric matrix $\theta_{\mu\nu}$ [31]. However, the canonical coordinate algebra and the Lorentz algebra are inconsistent with each other. Therefore, one needs to use Lie-algebra noncommutative spaces in order to generalized the commutative relativistic fluid to noncommutative Lorentz covariant models [32]. This was done in [33] where the action functional of the noncommutative fluid with the deformed Poincaré invariance was obtained for a general class of noncommutative spaces that are realizations of the Snyder algebra [1]. The realization method was developed in [34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Similar ideas with the realization method were presented in [44, 45, 46].

The commutative coordinates in the Snyder space $\mathcal{S}$ are the Lie generators of $so(1,4)/so(1,3)$. The degrees of freedom of the noncommutative fluid in $\mathcal{S}$ defined in [33] are the natural generalization of the commutative fluid potentials in the Clebsch parametrization. These belong to the set of functions over the Snyder space $\mathcal{F}(\mathcal{S})$ which can be endowed with the star-product and the co-product constructed in [47, 48]. The algebra $\mathcal{F}(\mathcal{S})$ is isomorphic to the deformed algebra over the Minkowski space-time $(C^{\infty}(\mathcal{M}), \star)$. Since the star-product is nonassociative and noncommutative and the momenta associated to the coordinates do not form a Lie group, understanding the dynamics of the noncommutative fluid in the Snyder space turns to be a very challenging problem. The reason is that the noncommutative fluid equations involve an infinite number of derivatives of the fluid potentials which are multiplied in a nonassociative way. This property makes the equations difficult to analyse in the general case. However, if the deformation parameter of the deformed Poincaré algebra $s = l_s^2$, where l_s is the typical length scale of the noncommutative space is small compared to unity, one can attempt to expand the star-product in powers of s. This is certainly the case if the noncommutative structure is the structure of the physical space-time since the phenomenological data and the theoretical arguments suggest that l_s is of the order of the Planck scale. Therefore, one can attempt to study the equations of motion of the noncommutative fluid perturbatively in s.

In this paper, we are going to develop a perturbative method to study the noncommutative fluid dynamics in the Snyder space. This method is based on the expansion in powers of s of the relevant objects from the algebra $(C^{\infty}(\mathcal{M}), \star)$ [42]. In particular, we are going to find solutions to the equations of motion of the fluid density current and potentials, respectively, in the linear approximation in the noncommutative parameter s. The dependence on s is encoded in the function on two momenta that determines the star-product and the co-product of the deformed Poincaré algebra and the anti-pode of its co-algebra. By truncating the power expansion at first order in s, the noncommutative fluid equations are reduced to a system of coupled linear partial differential equations. We are able to show that these equations admit monocromatic plane wave solutions for the density current j^{μ} and α and β fluid potentials, respectively and a linear solution for θ potential.

The paper is organized as follows. In Section 2, we review the noncommutative fluid in the Snyder space obtained in [33]. The perturbative expansion of the model and its first order equations of motion are obtained in Section 3. In Section 4, we show that the equations of motion admit solutions that are monocromatic plane waves of the fluid potentials. These solutions are characterized by the fact that the scalar product of divergence of the potentials α and β with the current density j^{μ} in the Minkowski space-time is zero. Also, we calculate the energy-momentum tensor for these solutions. In the last section we discuss the properties of the solutions obtained previously. For completeness, we present in the Appendix A the analytic expression of the energy-momentum tensor at first order in s which is too large to be

included in the body of the paper with any direct benefit.

2 Noncommutative fluid in the Snyder space

In this section we are going to review the noncommutative fluid in the Snyder space from [33] and establish our notations. The Snyder-like geometries can be viewed as realizations of the Snyder algebra which, at its turn, is a deformation of the algebra $so(1,3)$ with the deformation parameter $s = l_s^2$

$$[\tilde{x}_\mu, \tilde{x}_\nu] = sM_{\mu\nu}, \tag{1}$$

$$[p_\mu, p_\nu] = 0, \tag{2}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}, \tag{3}$$

$$[M_{\mu\nu}, \tilde{x}_\rho] = \eta_{\nu\rho}\tilde{x}_\mu - \eta_{\mu\rho}\tilde{x}_\nu, \tag{4}$$

$$[M_{\mu\nu}, p_\rho] = \eta_{\nu\rho}p_\mu - \eta_{\mu\rho}p_\nu, \tag{5}$$

where $\mu, \nu = \overline{0,3}$ and l_s has the dimension of length. The generators $M_{\mu\nu}$ can be expressed in terms of the commutative coordinates and momenta $\{x_\mu, p_\nu\}$ of the Minkowski space-time $\mathcal{M}$ in the usual fashion $M_{\mu\nu} = i(x_\mu p_\nu - x_\nu p_\mu)$. Many noncommutative spaces compatible with the Lorentz symmetry can be obtained by realizing geometrically the Snyder algebra (1)-(5) through the so called realization method. In these spaces, the generators $\tilde{x}_\mu$ are interpreted as noncommutative position operators associated to the sites of a lattice of typical length l_s. The closure of the commutators over $so(1,3)$ is equivalent to the statement that the lattice space is compatible with the Lorentz symmetry [1]. The functions $\tilde{x}_\mu(x,p)$ and their commutation relations with the generators p_μ are not determined by the Snyder algebra [39]. The realization method allows one to construct noncommutative spaces in which the functions $\tilde{x}_\mu(x,p)$ are momentum dependent rescalings of the coordinates

$$\tilde{x}_\mu(x,p) = \Phi_{\mu\nu}(s;p)x_\nu. \tag{6}$$

It can be shown that the smooth functions take the form $\Phi_{\mu\nu}(s;p) = \Phi_{\mu\nu}\left[\varphi(s;p)\right]$ such that

$$\tilde{x}_\mu(x,p) = x_\mu\varphi(A) + s\,\langle xp\rangle\, p_\mu \left[1 + 2\frac{d\varphi(A)}{dA}\right]\left[\varphi(A) - 2A\frac{d\varphi(A)}{dA}\right]^{-1}, \tag{7}$$

where $A = s\eta^{\mu\nu}p_\mu p_\nu$ and the commutative scalar product is denoted by $\langle ab\rangle = \eta^{\mu\nu}a_\mu b_\nu$. From the above equations, one can see that the Snyder geometry is a non-canonical deformation of the commutative phase space of coordinates $\{x_\mu, p_\mu\}$. The realization formalism allows one to work simultaneously with various noncommutative spaces which are characterized by different functions $\varphi(A)$. For example, the Weyl, the Snyder and the Maggiore noncommutative space-times can be obtaining by choosing $\varphi(A) = \sqrt{A}\cot(A)$, $\varphi(A) = 1$ and $\varphi(A) = \sqrt{1 - sp^2}$, respectively. Interpolations among these spaces are also possible [42].

The noncommutative fluid constructed in [33] generalizes the perfect relativistic fluid models in the Clebsch parametrization. The dynamics of the commutative fluid can be derived from an action functional $S[\phi(x)]$ that depends on the density current and three real fluid potentials $\phi(x) = \{j^\mu(x), \theta(x), \alpha(x), \beta(x)\}$ by applying field theoretical methods [20]. The action of the noncommutative fluid is determined by a correspondence principle that constraints the possible noncommutative functionals such that the equations of the relativistic fluid are obtained in the commutative limit

$$\lim_{s\to 0} S_s[\tilde{\phi}(\tilde{x})] = S[\phi(x)]. \tag{8}$$

It follows that the fluid potentials must be generalized to the functions $\tilde{\phi}(\tilde{x}) = \{\tilde{j}^{\mu}(\tilde{x}), \tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})\}$ from $\mathcal{F}(\mathcal{S})$ that should be identified with the degrees of freedom of the noncommutative fluid. It is useful to map the noncommutative and nonassociative algebra $\mathcal{F}(\mathcal{S})$ into the deformed algebra of the Minkowski space-time $(C^{\infty}(\mathcal{M}), \star)$ as follows. If $\tilde{\phi}(\tilde{x})$ is a noncommutative function and $\mathbf{1}$ is the identity element of the algebra of commutative functions over x_{μ} then

$$\tilde{\phi}(\tilde{x}) \triangleright \mathbf{1} = \psi(x), \tag{9}$$

where $\psi(x)$, in general, differs from $\phi(x)$. The star-product, the co-product and the anti-pode S of the Poincaré co-algebra are defined by the following relations

$$e^{i\left\langle K_1^{-1}(k_1)\tilde{x}\right\rangle} \star e^{i\left\langle K_2^{-1}(k_2)\tilde{x}\right\rangle} = e^{i\left\langle D^{(2)}(k_2,k_1)x\right\rangle}, \tag{10}$$

$$\triangle p_{\mu} = D_{\mu}^{(2)}(p \otimes \mathbf{1}, \mathbf{1} \otimes p), \tag{11}$$

$$D_{\mu}^{(2)}\left(g, S(g)\right) = 0, \tag{12}$$

for any element g of the deformed Poincaré group. Here, the non-commutative functions $e^{i\langle k\tilde{x}\rangle}$ depend on the deformed momentum $K_{\mu} = K_{\mu}(k)$ and are defined by the following relation

$$e^{i\langle k\tilde{x}\rangle} \triangleright \mathbf{1} = e^{i\langle K\tilde{x}\rangle}. \tag{13}$$

It follows from the equations (10) - (12) that the two-functions $D^{(2)}(k_2, k_1)$ determine completely the algebraic structure of the deformed algebra. By choosing the differential representation of the generators $p_{\mu} = -i\partial_{\mu}$ the star-product can be written as [37]

$$\left(f \star g\right)(x) = \lim_{y \to x} \lim_{z \to x} \exp\left[i \left\langle \left(D^{(2)}(p_y, p_z) - p_y - p_z\right) x \right\rangle\right]. \tag{14}$$

As was shown in [33] the action of the noncommutative fluid that obeys the correspondence principle (8) has the following form

$$\begin{aligned} S_s\left[j^{\mu}(x), \theta(x), \alpha(x), \beta(x)\right] &= \int d^4x \tilde{\mathcal{L}}[\tilde{\theta}(\tilde{x}), \tilde{\alpha}(\tilde{x}), \tilde{\beta}(\tilde{x})] \triangleright \mathbf{1} \\ &= \int d^4x \left[-j^{\mu}(x) \star \left[\partial_{\mu}\theta(x) + \alpha(x) \star \partial_{\mu}\beta(x)\right] - f_s\left(\sqrt{-j^{\mu}(x) \star j_{\mu}(x)}\right)\right], \end{aligned} \tag{15}$$

where the last equality defines a functional over $(C^{\infty}(\mathcal{M}), \star)$. The function f_s from $(C^{\infty}(\mathcal{M}), \star)$ is the image under the map (9) of an arbitrary function $\tilde{f}$ from $\mathcal{F}(\mathcal{S})$. It follows that the action (15) describes a class of noncommutative fluids parametrized by f_s for any given value of s.

In general, due to the lack of phenomenological information about the noncommutative fluid, it is difficult to define the relevant physical quantities such as the energy and momentum densities of the fluid. The functional approach to the noncommutative fluid has the advantage of making the definition of these quantities conceptually simpler, although their computation is difficultated by the nonassociativity of the star-product. The energy and momentum can be defined by the variation of the action (15) under $\delta_{\varepsilon}x_{\mu}$ that can be obtained from the deformed Poincaré transformations

$$\delta_{\nu}x_{\mu} = i\left(\eta_{\nu\mu} + sp_{\nu}p_{\mu}\varphi^{-1}(A)\left[1 + 2\frac{d\varphi(A)}{dA}\right]\left[\varphi(A) - 2A\frac{d\varphi(A)}{dA}\right]^{-1}\right), \tag{16}$$

$$\begin{aligned} \delta_{\rho\sigma}x_{\mu} = &\left[\eta_{\rho\sigma}\left(x_{\sigma} + s\left\langle xp\right\rangle p_{\sigma}\right)\varphi^{-1}(A)\left[1 + 2\frac{d\varphi(A)}{dA}\right]\left[\varphi(A) - 2A\frac{d\varphi(A)}{dA}\right]^{-1}\right] \\ &- \left[\eta_{\sigma\mu}\left(x_{\rho} + s\left\langle xp\right\rangle p_{\rho}\right)\varphi^{-1}(A)\left[1 + 2\frac{d\varphi(A)}{dA}\right]\left[\varphi(A) - 2A\frac{d\varphi(A)}{dA}\right]^{-1}\right]. \end{aligned} \tag{17}$$

It was shown in [33] that the variation of the action under the deformed translations results in the following energy-momentum tensor

$$T^{\mu}_{\nu} = \Theta^{\mu}_{\sigma}(\phi) - \mathcal{L}\eta^{\mu}_{\nu}, \tag{18}$$

where $\Theta^{\mu\nu}(\phi)$ is a functional of the fluid potentials and their derivatives up to the third order. An outstandingly difficult problem created by the noncommutativity and the nonassociativity of the star-product is to determine and to solve the equations of motion which, in general, form a system of nonlinear partial differential equations of arbitrarily high order and to obtain an analytic formula for T^{μ}_{ν}. This task becomes tractable at finite order in the powers of the noncommutative parameter s.

3 Lower order expansion in s

The analysis performed in the previous section has suggested that one should truncate the action at finite order in powers of s in order to compute the physical quantities of the noncommutative fluid. If the noncommutative parameter is of the order of the Planck constant, it is justified to allow only the terms linear in s. Another reason for which one should study the terms of the action (15) corresponding to the finite order in s is the following. Due to the nonassociativity of the star-product (14), the highest order of the derivatives of the fluid potentials from $S_s\left[j^{\mu}(x), \theta(x), \alpha(x), \beta(x)\right]$ is infinite. Therefore, the Euler-Lagrange equations of motion cannot be determined for an arbitrary value of s. Moreover, since the theory has just a limited number of symmetries, the equations of motion are not integrable. In this section, we are going to determine the equations of motion and the variation of the action under the deformed Poincaré transformations at the first order in s.

3.1 First order action and equations of motion

The nonassociative exponential from the star-product contains infinitely many derivatives of the fluid potentials. Therefore, the truncation of the star-product to some finite order in the powers of s is needed. The s dependence of the $\star$-product is encoded in the two-functions $D^{(2)}_{\mu}(k_1, k_2)$ [36]

$$D^{(2)}_{\mu}(k_1, k_2) = \sum_{n=1}^{\infty} s^n D^{(2)n}_{\mu}(k_1, k_2). \tag{19}$$

In order to obtain the linearized action in s, we consider only the first two terms from the above series

$$D^{(2)0}_{\mu}(k_1, k_2) = k_{1,\mu} + k_{2,\mu}\,, \tag{20}$$

$$D^{(2)1}_{\mu}(k_1, k_2) = A(k_1, k_2) k_{1,\mu} + B(k_1, k_2) k_{2,\mu}\,, \tag{21}$$

where the functions $A(k_1, k_2)$ and $B(k_1, k_2)$ have the following form

$$A(k_1, k_2) = c\left(k_2^2 + 2k_1k_2\right), \tag{22}$$

$$B(k_1, k_2) = \left(c - \frac{1}{2}\right) k_1^2 + \left(2c - \frac{1}{2}\right) k_1 k_2, \tag{23}$$

$$c = \frac{2c_1 + 1}{2}. \tag{24}$$

The real constant c_1 is realization dependent and has the following values: $c_1 = -1/2$ for the Maggiore, $c_1 = 0$ for the Snyder and $c_1 = -1/3$ for the Weyl realizations, respectively. The first order action can be obtained by linearizing simultaneously the star-product and the two-functions with respect to s. Some algebra shows that the linearized action takes the following form

$$
\begin{aligned}
S_s = & -\int d^4x \left\{ j^\mu(x)\partial_\mu\theta(x) + ix^\mu \left[K^s_\mu(y,z) j^\nu(y)\partial_\nu\theta(z)\right]|_{y=z=x} \right\} \\
& -\int d^4x \left\{ j^\mu(x)\alpha(x)\partial_\mu\beta(x) + ix^\mu \left[K^s_\mu(w,x) j^\nu(w)\alpha(x)\partial_\nu\beta(x)\right]|_{w=x} \right\} \\
& - i\int d^4x j^\nu(x) x^\mu \left[K^s_\mu(y,z) j^\nu(y)\partial_\nu\theta(z)\right] \alpha(y)\partial_\nu\beta(z)|_{y=z=x} \\
& + \int d^4x x^\mu x^\rho K^s_\mu(w,x) K^s_\rho(y,z) j^\nu(w)\alpha(y)\partial_\nu\beta(z)|_{w=y=z=x} \\
& + s\int d^4x x^\mu x^\rho \left[D^{(2)0}_\mu(-i\partial^w_\mu, -i\partial^z_\mu) + i\partial^w_\mu + i\partial^z_\mu \right] D^{(2)1}_\mu(-i\partial^y_\mu, -i\partial^z_\mu) j^\nu(w)\alpha(y)\partial_\nu\beta(z)|_{w=y=z=x} \\
& + \int d^4x f\left(- \left(1 + ix^\mu \left[K^s_\mu(y,z) j^\nu(y) j_\nu(z)\right]|_{y=z=x}\right)^{1/2}\right), \qquad (25)
\end{aligned}
$$

where we have used the momentum representation $k_\mu = -i\partial_\mu$ and the following notation

$$
K^s_\mu(y,z) = D^{(2)0}_\mu(-i\partial^y_\mu, -i\partial^z_\mu) + sD^{(2)1}_\mu(-i\partial^y_\mu, -i\partial^z_\mu) + i\partial^y_\mu + i\partial^z_\mu. \qquad (26)
$$

The inspection of the action (25) shows that the highest order of the derivatives of the fluid potentials is three. Therefore, the equations of motion take the general form

$$
\frac{\delta S_s}{\delta\phi} = \frac{\partial \mathcal{L}_s}{\partial\phi} - \partial_\mu \frac{\partial \mathcal{L}_s}{\partial(\partial_\mu\phi)} + \partial^2_{\mu\nu}\frac{\partial \mathcal{L}_s}{\partial(\partial^2_{\mu\nu}\phi)} - \partial^3_{\mu\nu\rho}\frac{\partial \mathcal{L}_s}{\partial(\partial^3_{\mu\nu\rho}\phi)} = 0. \qquad (27)
$$

By direct and lenghty calculations one can show that the equation of motion of j^μ has the following form

$$
\begin{aligned}
& -\partial_\mu\theta - \alpha\partial_\mu\beta + \partial_\nu\left(\frac{f'}{\rho_0}\right) x^\nu j_\mu + \frac{5}{2}s\left[\partial^2\partial_\mu\theta + \partial^2\left(\alpha\partial_\mu\beta\right)\right] \\
& + \frac{1}{2}sx^\nu\left[\partial^2\alpha\partial_\nu\partial_\mu\beta + 2\partial_\rho\alpha\partial^\rho\partial_\nu\partial_\mu\beta + \partial_\rho\partial_\nu\alpha\partial^\rho\partial_\mu\beta\right] \\
& + is\partial_\nu\left[\frac{f'}{2\rho_0}\left[\left(2c - \frac{1}{2}\right)\delta^{\nu\sigma}x_\sigma\partial^2 j_\mu + \left(4c - \frac{1}{2}\right)x_\sigma\partial^\sigma\partial^\nu j_\mu\right]\right] \\
& + is\partial^2_{\nu\omega}\left[\frac{f'}{2\rho_0}x_\sigma\left[\left(2c - \frac{1}{2}\right)\eta^{\omega\nu}\partial^\sigma j_\mu + \left(4c - \frac{1}{2}\right)\delta^{\omega\sigma}\partial^\nu j_\mu\right]\right] = 0. \qquad (28)
\end{aligned}
$$

Here, f' represents the derivative of f with respect to its argument $\rho_0 = \sqrt{-j^\mu(x) \star j_\mu(x)}$. In a similar way, one can derive the equation of motion for the fluid potential θ

$$
\partial_\mu j^\mu + \frac{1}{2}s\partial^2\partial_\mu j^\mu = 0. \qquad (29)
$$

The equation of motion of the potential α is given by the following relation

$$
\begin{aligned}
& -j^\mu\partial_\mu\beta + s\left(\partial^2 j^\mu\partial_\mu\beta + 2\partial_\mu j^\nu\partial^\mu\partial_\nu\beta + j^\mu\partial^2\partial_\mu\beta\right) \\
& + \frac{s}{2}x^\mu\left(\partial^2 j^\nu\partial_\mu\partial_\nu\beta + \partial_\mu j^\nu\partial^2\partial_\nu\beta + \partial_\mu\partial_\nu j^\sigma\partial^\nu\partial_\sigma\beta + \partial_\nu j^\sigma\partial_\mu\partial^\nu\partial_\sigma\beta\right) = 0. \qquad (30)
\end{aligned}
$$

Finally, the equation of motion of the potential β derived from (27) takes the following form

$$\partial_\mu\left(j^\mu\alpha\right)+s\left(3\partial_\mu j^\mu\partial^2\alpha+4\partial_\mu\partial_\nu j^\mu\partial^\mu\alpha+\frac{5}{2}\partial_\mu j^\nu\partial_\nu\partial^\mu\alpha+\partial^2 j^\mu\partial_\mu\alpha+\partial^2\partial_\mu j^\mu\alpha+j^\mu\partial^2\partial_\mu\alpha\right)=0. \tag{31}$$

We note that the zeroth order terms of the equation of motion (28) differ from the corresponding ones from the commutative equation of the current j^μ by a term proportional to $x^\nu j_\mu$ which can be cancelled by choosing proper boundary conditions. The equation (29) shows that the current density is not conserved in the noncommutative case which is a consequence of a lack of translation symmetry. The violation of the translation invariance has been previously analysed in [49]. Nevertheless, the conservation is restored in the commutative limit.

The lowest order expansion of the energy-momentum tensor can be computed by expanding T^μ_ν from the equation (18) in powers of s. The result is given by a not very illuminating formula which, for completeness, is presented in the Appendix A.

4 Plane waves in noncommutative fluids in Snyder space

The equations of motion (28), (29), (30) and (31) obtained in the previous section describe the dynamics of the noncommutative fluid at first order in s. Therefore, in order to understand better the dynamics, it is important to look for analytic solutions to the system (28) - (31). In this section, we are going to determine a particular set of solutions that are monocromatic waves of fluid potentials. To this end, we observe that the equations (29), (30) and (31) form a subsystem of coupled linear equations in each argument that involve the fluid potentials $j^\mu(x)$, $\alpha(x)$ and $\beta(x)$ which belong to the algebra $(C^\infty(\mathcal{M}),\langle\cdot\rangle)$ with the usual commutative product. This particular structure of the equations of motion suggests that one could solve them by trying to determine the solution to one of the equations and then using it in the coefficients of the rest of the equations.

The natural choice is to solve firstly (29) which is independent of the potentials $\alpha(x)$ and $\beta(x)$ and is second order and linear. By a simple field redefinition which lowers the degree of the equation by one the equation (29) is equivalent with the Klein-Gordon equation with the mass parameter

$$m_\phi^2=\frac{2}{s}. \tag{32}$$

It follows that the equation of motion of the current density admits plane wave solutions independent of the other fluid potentials

$$j^\mu_\pm(x)=\mp iA_\pm\frac{k^\mu}{k^2}e^{\pm i\langle kx\rangle}, \tag{33}$$

where $A_\pm$ is an arbitrary constant that can be taken one and $k^2=\langle kk\rangle$. As can be seen from the equation (30), the solutions $j^\mu_\pm(x)$ enter as coefficients in the equation of motion of the potential $\beta(x)$. One can also lower the degree of (30) by one and after somewhat lengthy but straightforward calculations, one can show that (30) has plane wave solutions $\beta^\pm(x)$ which have the property that the scalar products between their gradients with the current density in the Minkowski space-time is zero

$$\left\langle j(x)\partial\beta^\pm(x)\right\rangle=0. \tag{34}$$

The explicit form of the $\beta^{\pm}$ - waves is given by the following equation

$$\beta^{\pm}(x) = B^{\pm} e^{\mp i \langle kx \rangle}, \tag{35}$$

where $B^{\pm}$ are arbitrary constants. The $\beta^{\pm}$ - waves move in opposed direction to $j^{\mu}_{\pm}$ - waves. By following the same steps as above, it is possible to show that the equation (31) has plane wave solutions of gradients that satisfy the same condition

$$\langle j(x) \partial \alpha^{\pm}(x) \rangle = 0. \tag{36}$$

The $\alpha^{\pm}$ - wave solutions have the following form

$$\alpha^{\pm}(x) = C^{\pm} e^{\pm i \langle kx \rangle}, \tag{37}$$

where $C^{\pm}$ are arbitrary constants.

Let us focus now on the equation (28) that describes the dynamics of the potential θ as a function of the current density and the other two fluid potentials. In order to find its solution, one needs to plug the previous wave solutions (33), (35) and (37) into (28). After doing that and after some algebraic manipulations, we obtain the following equation

$$\begin{aligned}\left(\partial^2 - \frac{2}{5s}\right) \partial_\mu \theta^{\pm} = & \pm \frac{2iA_{\pm}}{5s} \frac{k_\mu}{k^2} x^\nu \partial_\nu \left(\frac{f'}{\rho_0}\right) e^{\pm i \langle kx \rangle} \\ & - \frac{2i}{5} \left[\left(\Omega^{\pm}_{\mu}(k) \eta^{\sigma}_{\nu} + \Lambda^{\sigma\nu}_{\pm\mu}(k)\right) \partial_\sigma + \left(\Gamma^{\omega\sigma\nu}_{\pm\mu}(k) + \Upsilon^{\omega\sigma\nu}_{\pm\mu}(k)\right) \partial^2_{\sigma\omega} \right] \left(\frac{f'}{\rho_0} x^\nu e^{\pm i \langle kx \rangle}\right) \\ & - \frac{2i}{5s} D_{\pm} k_\mu. \end{aligned} \tag{38}$$

Here, we have introduced the following notations for the momenta depending coefficients in an arbitrary realization

$$\begin{aligned} \Omega^{\pm}_{\mu}(k) &= \pm \frac{i}{2} \left(2c - \frac{1}{2}\right) A_{\pm} k_\mu, \\ \Lambda^{\nu\sigma}_{\pm\mu}(k) &= \pm \frac{i}{2} \left(4c - \frac{1}{2}\right) A_{\pm} \frac{k_\mu k^\nu k^\sigma}{k^2}, \end{aligned} \tag{39}$$

$$\Gamma^{\omega\nu\sigma}_{\pm\mu}(k) = \left(2c - \frac{1}{2}\right) A_{\pm} \eta^{\omega\nu} \frac{k_\mu k^\sigma}{k^2}, \tag{40}$$

$$\Upsilon^{\omega\sigma\nu}_{\pm\mu}(k) = \left(4c - \frac{1}{2}\right) A_{\pm} \delta^{\omega\sigma} \frac{k_\mu k^\nu}{k^2}. \tag{41}$$

We note that while $j^{\mu}_{\pm}$ -, $\alpha^{\pm}$ - and $\beta^{\pm}$ - waves, respectively, are common to all noncommutative fluids from the class described by the linearized action (25), the potentials $\theta^{\pm}(x)$ depend on the details of each particular model. As was mentioned in the previous sections, a model can be specified by choosing the arbitrary function $f(\rho_0)$ and the fluid density function $\rho_0(x)$.

A simple class of models is given by $f(\rho_0) = \lambda \rho_0^2/2$ where λ is a real parameter and ρ_0 is an arbitrary function. This type of commutative fluids has been discussed in [23] in the Kähler parametrization, while in [29] it was shown that they can be quantized using canonical methods. Then the equation (38) takes the following simpler form

$$\left(\partial^2 - \frac{2}{5s}\right) \partial_\mu \theta^{\pm} = -\frac{2i}{5s} D_{\pm} k_\mu, \tag{42}$$

The above equation can be integrated by standard methods or the solutions can be simply guessed. In either way, we find that the solutions $\theta^{\pm}$ that correspond to $j^{\mu}_{\pm}$ -, $\alpha^{\pm}$ - and $\beta^{\pm}$-waves are linear functions on x

$$\theta^{\pm}(x) = E^{\pm} \pm iD_{\pm} \langle kx \rangle , \tag{43}$$

where $E^{\pm}$ are arbitrary integration constants. The fact that the $\theta^{\pm}$ - potentials are linear implies that they contribute by constant pieces to the fluid velocity $v^{\pm}_{\mu} \sim \partial_{\mu}\theta^{\pm} + \cdots$ in the commutative limit.

4.1 The energy-momentum tensor

The energy-momentum tensor of the noncommutative fluid has the general form given by the equation (18). By expanding in powers of s and retaining only the linear terms we obtain the linearized energy-momentum tensor $T^{\mu}_{\nu}(j,\theta,\alpha,\beta)$ given by the equation (46) from the Appendix. From it, one can calculate the energy-momentum tensor of the wave potentials $T^{\mu}_{\nu}(j_{\pm},\theta^{\pm},\alpha^{\pm},\beta^{\pm})$. The computations are somewhat lenghty but elementary. The result takes the following from

$$\begin{aligned} T^{\mu}_{\nu}(j^{\pm},\theta^{\pm},\alpha^{\pm},\beta^{\pm}) = {} & A_{\pm}\left(D^{\pm} - C^{\pm}B^{\pm}\right) e^{\pm i\langle kx\rangle}\left(\eta^{\mu}_{\nu} - \frac{k^{\mu}k_{\nu}}{k^2}\right) \\ & + \frac{s\lambda}{4}\left(A_{\pm}\right)^2 \frac{e^{\pm 2i\langle kx\rangle}}{k^2}\left[\left(\frac{7}{2} \pm 2i\langle kx\rangle\right)k^{\mu}k_{\nu} \pm 2ik^2x^{\mu}k_{\nu}\right] \\ & + \frac{s}{2}A_{\pm}D^{\pm}e^{\pm i\langle kx\rangle}\left[\left(1 \mp i\langle kx\rangle\right)k^{\mu}k_{\nu} \pm ik^2\langle kx\rangle\eta^{\mu}_{\nu}\right] \\ & \mp \frac{is}{2}\frac{A_{\pm}B^{\pm}C^{\pm}}{k^2}e^{\pm i\langle kx\rangle}\left[-\left(8 \mp 7i\right)k^2\langle kx\rangle k^{\mu}k_{\nu} + \left(3 \pm 5i\right)k^2k^{\mu}k_{\nu} + \left(2 \mp i\right)k^4x^{\mu}k_{\nu}\right]. \end{aligned} \tag{44}$$

Note that the energy-momentum tensor of the wave fluids in the noncommutative fluid model studied above is complex as it is in the case of spinor fields and $N = 1$ chiral supergravity in the commutative space-time. As can be seen from the equation (44), the imaginary part is a consequence of the noncommutativity of space-time.

5 Concluding remarks

In this paper we have proposed a perturbative method to analyse the dynamics of the noncommutative fluid in the Snyder space based on the expansion of the star-product, co-product and the anti-pode in powers of the noncommutative parameter s. At first order in s the equations of motion of the fluid density and the fluid potentials form a system of linear partial differential equations. We have determined a class of perturbative analytic solutions of these equations that describe monocromatic plane waves of j^{μ} and α and β potentials. For the class of fluids characterized by $f(\rho_0) = \lambda\rho_0^2/2$ the equation of motion of the potential θ admits a linear solution. The energy-momentum tensor of the plane waves receives a complex correction at first order in s. The noncommutative parameter determines at first order the dispersion relations of the monocromatic wave potentials as follows

$$k_j^2 - \frac{2}{s} = 0, \qquad k_{\alpha}^2 = \frac{1-s}{3s}, \qquad k_{\beta}^2 = 0, \tag{45}$$

which can be used to determine the on-shell solutions and energy-momentum tensor. The geometry of the waves is such that the gradients of α and β potentials is normal to the direction of j^{μ}. However, the plane waves do not create vorticity in the linear approximation of the noncommutative fluid regardless the choice of the parameters f and ρ_0 which is an expected result since the vorticity is not a well defined concept in the Snyder space. Nevertheless, the model can still produce in the commutative limit $s \to 0$ a rotational fluid.

The results presented in this paper provide a new insight in the dynamics of the noncommutative fluids. They are interesting by themselves, as well as for providing new models of noncommutative field theories. From this point of view, it is of interest the explicit calculation of the energy-momentum tensor in the linear approximation. A similar approach to the perturbative treatment of the noncommutative fields was undertaken in [42] where the corrections to the scalar field dynamics have been obtained at first order. We are going to report on other aspects of this model in future.

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A Energy - momentum tensor

The general properties of the energy-momentum tensor in the Snyder space are defined by the invariance of the action (15) under the deformed translations (16). In order to obtain the explicit for of T^{μ}_{ρ} we calculate the variation (15) at first order in s. Since it is expected that the conservation of the energy-momentum tensor under translations hold on-shell, we use the equations of motion (28) - (31). The computations are somewhat lenghty but quite straightforward. The result is given by the following equation

$$T^{\mu}_{\rho}(j,\theta,\alpha,\beta) = T^{\mu}_{0,\rho}(j,\theta,\alpha,\beta) + T^{\mu}_{1,\rho}(j,j) + T^{\mu}_{1,\rho}(j,\theta) + T^{\mu}_{1,\rho}(j,\alpha,\beta), \tag{46}$$

where

$$
\begin{aligned}
T^{\mu}_{0,\rho}(j,\theta,\alpha,\beta) &= -j^{\mu}\partial_{\rho}\theta - j^{\mu}\alpha\partial_{\rho}\beta + (j^{\nu}\partial_{\nu}\theta + j^{\nu}\alpha\partial_{\nu}\beta)\,\eta^{\mu}_{\rho},\\
T^{\mu}_{1,\rho}(j,j) &= \frac{s}{2}\Big[-\frac{f'}{2\rho_0}x^{\lambda}\partial^{\mu}\partial_{\lambda}j_{\gamma}\partial_{\rho}j^{\gamma} + \partial_{\nu}[\frac{f'}{\rho_0}(x^{\mu}\partial^{\nu}j_{\gamma} + \frac{1}{2}x^{\nu}\partial^{\mu}j_{\gamma})]\partial_{\rho}j^{\gamma} + \partial^2_{\nu\sigma}(\frac{f'}{2\rho_0}x^{\sigma}\eta^{\mu\nu}j_{\gamma})\partial_{\rho}j^{\gamma}\\
&\quad + \frac{f'}{\rho_0}(x^{\mu}\partial^{\nu}j_{\gamma} + \frac{1}{2}x^{\nu}\partial^{\mu}j_{\gamma})\partial^2_{\nu\rho}j^{\gamma} + \frac{f'}{2\rho_0}x^{\sigma}\eta^{\mu\nu}j_{\gamma}\partial^3_{\nu\sigma\rho}j^{\gamma}\Big],\\
T^{\mu}_{1,\rho}(j,\theta) &= \frac{s}{2}\Big[x^{\lambda}\partial^2\partial_{\lambda}j^{\mu}\partial_{\rho}\theta - 2\partial_{\nu}(x^{\lambda}\partial_{\lambda}\partial^{\nu}j^{\mu})\partial_{\rho}\theta + \partial^2_{\nu\sigma}(x^{\nu}\partial^{\sigma}j^{\mu})\partial_{\rho}\theta\\
&\quad + 2x^{\lambda}\partial^{\mu}\partial_{\lambda}\partial_{\gamma}\theta\partial_{\rho}j^{\gamma} - 2\partial_{\nu}(x^{\nu}\partial^{\mu}\partial_{\gamma}\theta)\partial_{\rho}j^{\gamma} + \partial^{\mu}\partial_{\sigma}(x^{\sigma}\partial_{\gamma}\theta)\partial_{\rho}j^{\gamma}\\
&\quad + 2x^{\lambda}\partial_{\lambda}\partial^{\nu}j^{\mu}\partial^2_{\nu\rho}\theta + 2x^{\nu}\partial^{\mu}\partial_{\gamma}\theta\partial^2_{\nu\rho}j^{\gamma}\\
&\quad + x^{\nu}\partial^{\sigma}j^{\mu}\partial^3_{\nu\sigma\rho}\theta + x^{\sigma}\partial_{\gamma}\theta\partial^{\mu}\partial^2_{\sigma\rho}j^{\gamma} - \partial_{\nu}(x^{\nu}\partial^{\sigma}j^{\mu})\partial^2_{\rho\sigma}\theta - \partial^{\mu}(x^{\sigma}\partial_{\gamma}\theta)\partial^2_{\rho\sigma}j^{\gamma}\Big]\\
&\quad - \frac{s}{2}x^{\lambda}\left[\partial^2\partial_{\lambda}j^{\tau}\cdot\partial_{\tau}\theta + 2\partial_{\lambda}\partial_{\nu}j^{\tau}\cdot\partial^{\nu}\partial_{\tau}\theta + \partial_{\nu}j^{\tau}\cdot\partial^{\nu}\partial_{\lambda}\partial_{\tau}\theta\right]\eta^{\mu}_{\rho},\\
T^{\mu}_{1,\rho}(j,\alpha,\beta) &= \frac{s}{2}\Big[(j^{\omega}x^{\lambda}\partial^{\mu}\partial_{\lambda}\partial_{\omega}\beta + x^{\tau}\partial_{\omega}j^{\mu}\partial^{\omega}\partial_{\tau}\beta + x^{\tau}\partial_{\mu}j^{\nu}\partial_{\nu}\partial_{\tau}\beta + 2x^{\tau}\partial_{\tau}\partial^{\mu}j^{\nu}\partial_{\nu}\beta)\partial_{\rho}\alpha\\
&\quad - 2\partial_{\nu}(j^{\omega}x^{\mu}\partial^{\nu}\partial_{\omega}\beta + x^{\tau}\partial^{\nu}j^{\mu}\partial_{\tau}\beta)\partial_{\rho}\alpha + \partial^{\mu}\partial_{\sigma}(j^{\omega}x^{\sigma}\partial_{\omega}\beta)\partial_{\rho}\alpha\\
&\quad + (j^{\mu}x^{\lambda}\partial^2\partial_{\lambda}\alpha + x^{\mu}\partial_{\omega}j^{\lambda}\partial^{\omega}\partial_{\lambda}\alpha + x^{\tau}\partial^2\partial_{\tau}j^{\mu}\alpha + 2x^{\tau}\partial_{\tau}\partial_{\omega}j^{\mu}\partial^{\omega}\alpha)\partial_{\rho}\beta\\
&\quad - \partial_{\nu}[2j^{\mu}x^{\lambda}\partial_{\lambda}\partial^{\nu}\alpha + x^{\nu}(\partial^{\mu}j^{\lambda}\partial_{\lambda}\alpha + \partial_{\omega}j^{\mu}\partial^{\omega}\alpha) + 2x^{\tau}\partial_{\tau}\partial^{\nu}j^{\mu}\alpha]\partial_{\rho}\beta\\
&\quad + \partial^{\omega}\partial_{\sigma}(j^{\mu}x^{\sigma}\partial_{\omega}\alpha + x^{\sigma}\partial_{\omega}j^{\mu}\alpha)\partial_{\rho}\beta + x^{\tau}\partial_{\mu}\partial_{\gamma}(\alpha\partial_{\tau}\beta)\partial_{\rho}j^{\gamma}\\
&\quad - 2\partial_{\nu}(x^{\mu}\partial^{\nu}\alpha\partial_{\gamma}\beta + x^{\mu}\alpha\partial^{\nu}\partial_{\gamma}\beta)\partial_{\rho}j^{\gamma} + \partial^{\mu}\partial_{\sigma}(x^{\sigma}\alpha\partial_{\gamma}\beta)\partial_{\rho}j^{\gamma}\\
&\quad + (2j^{\omega}x^{\mu}\partial^{\nu}\partial_{\omega}\beta + x^{\tau}\partial^{\nu}j^{\mu}\partial_{\tau}\beta)\partial^2_{\nu\rho}\alpha\\
&\quad + [2j^{\mu}x^{\lambda}\partial_{\lambda}\partial^{\nu}\alpha + (\partial^{\mu}j^{\lambda}\partial_{\lambda}\alpha + \partial_{\omega}j^{\mu}\partial^{\omega}\alpha) + 2x^{\tau}\partial_{\tau}\partial^{\nu}j^{\mu}\alpha]\partial^2_{\nu\rho}\beta\\
&\quad + 2(x^{\mu}\partial^{\nu}\alpha\partial_{\gamma}\beta + x^{\mu}\alpha\partial^{\nu}\partial_{\gamma}\beta)\partial^2_{\nu\rho}j^{\gamma} - \partial^{\mu}(x^{\sigma}\alpha\partial_{\gamma}\beta)\partial^2_{\rho\sigma}j^{\gamma}\\
&\quad + j^{\omega}x^{\sigma}\partial_{\omega}\beta\partial^{\mu}\partial^2_{\sigma\rho}\alpha + (j^{\mu}x^{\sigma}\partial^{\nu}\alpha + x^{\sigma}\partial^{\nu}j^{\mu}\alpha)\partial^3_{\nu\sigma\rho}\beta + x^{\sigma}\alpha\partial_{\gamma}\beta\partial^{\mu}\partial^2_{\sigma\rho}j^{\gamma}\\
&\quad - \partial^{\mu}(j^{\omega}x^{\sigma}\partial_{\omega}\beta)\partial^2_{\rho\sigma}\alpha - \partial^{\omega}(j^{\mu}x^{\sigma}\partial_{\omega}\alpha + \frac{s}{2}x^{\sigma}\partial_{\omega}j^{\mu}\alpha)\partial^2_{\rho\sigma}\beta\Big]\\
&\quad - \frac{s}{2}j^{\nu}x^{\lambda}\left[\partial^2\partial_{\lambda}\alpha\cdot\partial_{\nu}\beta + 2\partial_{\lambda}\partial_{\tau}\alpha\cdot\partial^{\tau}\partial_{\nu}\beta + \partial_{\tau}\alpha\cdot\partial^{\tau}\partial_{\lambda}\partial_{\nu}\beta\right]\eta^{\mu}_{\rho}\\
&\quad - \frac{s}{2}x^{\tau}\left[\partial_{\omega}j^{\nu}\partial^{\omega}\partial_{\nu}(\alpha\partial_{\tau}\beta) + \partial^2\partial_{\tau}j^{\nu}\cdot\alpha\partial_{\nu}\beta + 2\partial_{\tau}\partial_{\omega}j^{\nu}\cdot\partial^{\omega}\alpha\cdot\partial_{\nu}\beta + 2\partial_{\tau}\partial_{\omega}j^{\nu}\cdot\alpha\cdot\partial^{\omega}\partial_{\nu}\beta\right]\eta^{\mu}_{\rho}\\
&\quad - f\left(\left(-j^{\tau}j_{\tau} + \frac{s}{2}x^{\lambda}\left[\partial^2\partial_{\lambda}j^{\tau}\cdot j_{\tau} + 2\partial_{\lambda}\partial_{\nu}j^{\tau}\cdot\partial^{\nu}j_{\tau} + \partial_{\nu}j^{\tau}\cdot\partial^{\nu}\partial_{\lambda}j_{\tau}\right]\right)^{\frac{1}{2}}\right)\eta^{\mu}_{\rho}
\end{aligned}
$$

The above equations represents the energy-momentum tensor of the noncommutative fluid in the Snyder space in the first order approximation. In the commutative limit $s \to 0$ the tensor T^{μ}_{ρ} coincides with the energy-momentum tensor of the relativistic perfect fluid. Also, T^{μ}_{ρ} is conserved under the on-shell translations

$$\partial_{\mu}T^{\mu}_{\rho} = 0. \tag{47}$$

We note that $T_{\mu\nu} = \eta_{\mu\rho}T^{\rho}_{\nu}$ does not have a definite symmetry. Actually, a symmetric energy-momentum tensor can be obtained by coupling the fluid with a c - number metric $g_{\mu\nu}$ and by deriving the action with respect to it (see [31]). However, by this procedure information about the noncommutative properties of the fluid could be lost due to the contraction between the antisymmetric components of the star-product and the metric.

References

[1] H. S. Snyder, Phys. Rev. **71**, 38 (1947).

[2] H. S. Snyder, Phys. Rev. **72**, 68 (1947).

[3] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001) [hep-th/0106048].

[4] R. J. Szabo, Phys. Rept. **378**, 207 (2003) [hep-th/0109162].

[5] R. Jackiw, S. Y. Pi and A. P. Polychronakos, Annals Phys. **301**, 157 (2002).

[6] R. Jackiw, arXiv:physics/0209108.

[7] R. Jackiw, Nucl. Phys. Proc. Suppl. **127**, 53 (2004).

[8] R. Jackiw, V. P. Nair, S. Y. Pi and A. P. Polychronakos, J. Phys. A **37**, R327 (2004).

[9] M. V. Marcial, A. C. R. Mendes, C. Neves, W. Oliveira, F. I. Takakura, Phys. Lett. **A374**, 3608-3613 (2010).

[10] S. Bahcall, L. Susskind, Int. J. Mod. Phys. **B5**, 2735-2750 (1991).

[11] L. Susskind, arXiv:hep-th/0101029.

[12] A. El Rhalami, E. M. Sahraoui and E. H. Saidi, JHEP **0205**, 004 (2002).

[13] J. L. F. Barbon and A. Paredes, Int. J. Mod. Phys. A **17**, 3589 (2002).

[14] J. L. F. Barbon and D. Gerber, Int. J. Mod. Phys. A **22**, 5287 (2007).

[15] A. P. Polychronakos, arXiv:0706.1095 [hep-th].

[16] S. A. Alavi, Chin. Phys. Lett. **23**, 10 (2006).

[17] A. De Felice, J. M. Gerard and T. Suyama, Phys. Rev. D **81**, 063527 (2010).

[18] S. A. Alavi, Prob. Atomic Sci. Technol. **3**, 301 (2007) [arXiv:0709.4163 [hep-th]].

[19] W. -H. Huang, JHEP **0908**, 102 (2009) [arXiv:0901.0614 [hep-th]].

[20] R. Jackiw, arXiv:physics/0010042.

[21] B. Carter, *Covariant Theory of Conductivity in Ideal Fluid or Solid Media*, em Anile, A. Anile e M. Choquet-Bruhat, eds., *Relativistic Fluid Dynamics*, aulas apresentadas na 1^a sessão de 1987 do Centro Internazionale Matematico Estivo (C.I.M.E.) em Noto, Italia, 25 de Maio 25 – 3 de June1987, vol. 1385 de Lecture Notes in Mathematics, 1–64, (Springer, Berlin, Germany; New York, U.S.A., 1989).

[22] R. Jackiw, [arxiv:physics/0010042]

[23] T. S. Nyawelo, J. W. van Holten and S. Groot Nibbelink, Phys. Rev. D **68**, 125006 (2003).

[24] R. Jackiw and A. P. Polychronakos, Phys. Rev. D **62**, 085019 (2000).

[25] D. Baleanu, Czech. J. Phys. **55**, 473 (2005).

[26] P. D. Jarvis and J. W. van Holten, Nucl. Phys. B **734**, 272 (2006).

[27] T. S. Nyawelo, Nucl. Phys. B **672**, 87 (2003).

[28] P. A. Grassi, A. Mezzalira and L. Sommovigo, arXiv:1107.2780 [hep-th].

[29] L. Holender, M. A. Santos and I. V. Vancea, Phys. Rev. D **77**, 045024 (2008).

[30] L. Holender, M. A. Santos and I. V. Vancea, Phys. Lett. A **376**, 1274 (2012).

[31] L. Holender, M. A. Santos, M. T. D. Orlando and I. V. Vancea, Phys. Rev. D **84**, 105024 (2011).

[32] G. Amelino-Camelia and M. Arzano, Phys. Rev. D **65**, 084044 (2002).

[33] M. C. B. Abdalla, L. Holender, M. A. Santos and I. V. Vancea, Phys. Rev. D **86**, 045019 (2012).

[34] L. Jonke and S. Meljanac, Phys. Lett. B **526**, 149 (2002).

[35] S. Meljanac and M. Stojic, Eur. Phys. J. C **47**, 531 (2006).

[36] S. Kresic-Juric, S. Meljanac and M. Stojic, Eur. Phys. J. C **51**, 229 (2007).

[37] S. Meljanac, A. Samsarov, M. Stojic and K. S. Gupta, Eur. Phys. J. C **53**, 295 (2008).

[38] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac and D. Meljanac, Phys. Rev. D **77**, 105010 (2008).

[39] M. V. Battisti and S. Meljanac, Phys. Rev. D **79**, 067505 (2009).

[40] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac and D. Meljanac, Phys. Rev. D **80**, 025014 (2009).

[41] S. Meljanac, D. Meljanac, A. Samsarov and M. Stojic, Mod. Phys. Lett. A **25**, 579 (2010).

[42] M. V. Battisti and S. Meljanac, Phys. Rev. D **82**, 024028 (2010).

[43] S. Meljanac, D. Meljanac, A. Samsarov and M. Stojic, Phys. Rev. D **83**, 065009 (2011).

[44] J. Lukierski, H. Ruegg and W. J. Zakrzewski, Annals Phys. **243**, 90 (1995).

[45] R. Banerjee, S. Kulkarni and S. Samanta, JHEP **0605**, 077 (2006) [hep-th/0602151].

[46] S. Ghosh and P. Pal, Phys. Rev. D **75**, 105021 (2007).

[47] F. Girelli and E. R. Livine, arXiv:0910.3107 [hep-th].

[48] F. Girelli and E. R. Livine, JHEP **1103**, 132 (2011).

[49] O. Bertolami and L. Guisado, JHEP **0312**, 013 (2003) [hep-th/0306176].